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An integral inequality concerning isotropic measures on the unit circle ☆

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Abstract

We prove a trigonometric integral inequality involving isotropic measures in the plane which can be applied to characterize the solution of extremal problems of convex bodies in $\mathbb{R}^2$ in terms of properties of measures. The methods used include new estimates of hypergeometric functions and some cancellation lemmas.

1. Introduction and notation

Let f be a positive, continuous function defined on the unit circle, $f:\mathbb{T}\to(0,\infty)$. Consider the function given by

$$F_{f,j}(a,\alpha)=F_j(a,\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta)\,d\theta}{(a^2\cos^2(\theta+\alpha)+a^{-2}\sin^2(\theta+\alpha))^{j/2}}$$

for any $a>0$ and any $\alpha, j\in\mathbb{R}$. The problem we are considering is to determine the extreme values of the function F_j. A simple computation gives a geometric interpretation of this problem and shows the motivation for it. Let $\rho_K(\cdot)$ be a radial function of a star-shaped body $K\subset\mathbb{R}^2$ with respect to the origin, i.e., $\rho_K(\theta)=\max\{\lambda\geqslant 0\colon\ \lambda(\cos\theta,\sin\theta)\in K\}$ for

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any $\theta \in \mathbb{T}$ (see, for example, [3,5]). If we consider for any $j \in \mathbb{R}$ the dual quermassintegral $\tilde{W}_j(K)$ given by

$$\tilde{W}_j(K) = \frac{1}{2}\int_{\mathbb{T}} \rho_K(\theta)^{2-j}\, d\theta$$

(see [3,4]), then $F_j(a,\alpha) = \tilde{W}_j(SK)$, where $S \in SL(2)$ is the linear transformation defined by

$$\begin{pmatrix} \pm a\cos\alpha & \mp a\sin\alpha \\ a^{-1}\sin\alpha & a^{-1}\cos\alpha \end{pmatrix}.$$

In [2], the authors study the problem of determining the positions of the convex body $L \subseteq \mathbb{R}^n$ for which

$$\tilde{W}_j(L) = \max \quad \text{or} \quad \min\bigl\{\tilde{W}_j(SL);\ S \in SL(n)\bigr\},$$

depending on the index j. In the particular case of the plane, this is actually the problem of computing extreme values for the function F_j. Since $SL(2)$ is a group, we can reduce the problem of finding the extreme values for F_j to finding necessary and sufficient conditions for F_j to attain its extreme value for $a = 1$ and $\alpha = 0$. If we use partial derivatives it is easy to check that

$$\frac{\partial F_j}{\partial a}(a = 1, \alpha = 0) = 0 \quad \Leftrightarrow \quad \hat{f}(2) = 0, \tag{1.1}$$

$$\frac{\partial F_j}{\partial \alpha}(a = 1, \alpha = 0) = 0 \quad \Leftrightarrow \quad \hat{f}(-2) = 0, \tag{1.2}$$

hence, if F_j attains its extreme value for $a = 1$ and $\alpha = 0$ then the Fourier coefficients $\hat{f}(\pm 2) = 0$. These conditions can be expressed in terms of isotropic measures. We recall that a measure μ on $\mathbb{T}$ is *isotropic* if and only if

$$\int_{\mathbb{T}} u_i u_k\, d\mu(\theta) = C\delta_{i,k},$$

$i, k \in \{1, 2\}$, where $(u_1, u_2) = (\cos\theta, \sin\theta)$. Hence the conditions (1.1) and (1.2) simply mean that the measure $f(\theta)\, d\theta$ is isotropic and as a consequence a necessary condition for F_j to attain its extreme value for $a = 1$ and $\alpha = 0$ is that $f(\theta)\, d\theta$ is isotropic.

The problem we are interested in is if the converse of the last assertion also holds, i.e., is it true that if $f(\theta)\, d\theta$ is isotropic then F_j attains the extreme value for $a = 1$ and $\alpha = 0$? More generally, it can be checked that if we take a Borel measure μ on $\mathbb{T}$ and we consider

$$F_{\mu,j}(a,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\mu(\theta)}{(a^2\cos^2(\theta+\alpha) + a^{-2}\sin^2(\theta+\alpha))^{j/2}},$$

a necessary condition in order that $F_{\mu,j}$ attains its extreme value for $a = 1$ and $\alpha = 0$ is that μ is isotropic. Hence we can ask if the reverse is also true.

For $j < 0$, the problem has an affirmative answer, since by using general properties of isotropic measures (see [2]) the following result can be proved.

Theorem 1.1. *Let μ be an isotropic probability on $\mathbb{T}$ then $F_{\mu,j}(a,\alpha) \geqslant F_{\mu,j}(1,0)$ for all $j < 0$, $0 < a$ and $\alpha \in \mathbb{R}$.*

For $j \geqslant 3$, there is a similar result for a particular kind of measure, and the following was proved very recently in [2] (by using convexity methods).

Theorem 1.2. *Let $f(\theta) = \rho_K(\theta)^{2-j}$, where K is a centrally symmetric convex body in $\mathbb{R}^2$. If $\hat{f}(\pm 2) = 0$, then $F_j(a,\alpha) \geqslant F_j(1,0)$ for all $j \geqslant 3$, $a > 0$ and $\alpha \in \mathbb{R}$.*

In the case $j \in (0,2)$, the problem we are dealing with is if an isotropic Borel measure on $\mathbb{T}$ satisfies the condition that

$$F_{\mu,j}(a,\alpha) \leqslant F_{\mu,j}(1,0). \tag{1.3}$$

Note that we cannot expect a result for general Borel probabilities in this case, as the following counterexample shows.

Example 1.3. If we consider the Borel measure $\mu = (1/4)(\delta_0 + \delta_{\pi/2} + \delta_\pi + \delta_{3\pi/2})$ and take $\alpha = 0$, it can be checked that

$$F_{\mu,j}(a,0) = \frac{1}{2}(a^{-j} + a^j) > 1$$

as a direct consequence of the arithmetic–geometric mean inequality. Consequently, one might think that we only should consider absolutely continuous measures $d\mu(\theta) = f(\theta)\,d\theta$, but a straightforward approximation argument ensures that for general C^∞ positive functions f or even for measures of the form $d\mu(\theta) = \rho_L^{2-i}(\theta)\,d\theta$, with L a general star body, the result is not true, so we have to restrict ourselves to a very particular case of absolutely continuous measures on $\mathbb{T}$.

Our main result is the following

Theorem 1.4. *Let $f : \mathbb{T} \to (0,\infty)$ be a continuous, 2π-periodic, positive function whose Fourier series is of the form*

$$f(\theta) \sim \sum_{k=0}^{\infty} B_{4k}\cos(4k\theta).$$

Suppose that the Fourier coefficients satisfy

(i) $|B_{4k}| \leqslant |B_8|$ *for all* $k \geqslant 2$ *and* $|B_4| < 0.070B_0$, $|B_8| < 0.022B_0$,
(ii) $\|f\|_\infty < 1.261B_0$.

Then $F_j(a,\alpha) \leqslant F_j(1,0)$ for all $a, \alpha \in \mathbb{R}$ and for all j close to 1.

The special form of the Fourier series is satisfied for (and is equivalent to) functions such that $f(\theta) = f(2\pi - \theta) = f(\pi - \theta) = f(\pi/2 - \theta)$ for all $\theta \in [0, 2\pi]$. In this case we will say that f has "*enough symmetries.*" A *geometric* example of this kind of function is

$(2-j)$-power of the radial function of a star body, symmetric with respect to the coordinate axes and to the bisectors of the quadrants.

In Section 2 we present the proof of Theorem 1.4, which uses sharp estimates for the hypergeometric functions that may be of independent interest.

In the final section we apply our main result to the extreme dual quermassintegral of convex bodies in the plane, in particular to the case $f(\cdot) = \rho_{B_1^2}(\cdot)$, where B_1^2 is the unit ball of ℓ_1^2, since its corresponding Fourier coefficients satisfy the conditions (i) and (ii) of Theorem 1.4. We prove a couple of lemmas for functions with "*enough symmetries*" which guarantee the conditions on the Fourier coefficients we need. Several subtleties for cancellations of Fourier coefficients appear there which are of interest in themselves.

2. The proof of the main theorem

In order to prove Theorem 1.4, we will combine two points of view. On the one hand we will give some general estimates for $F_j(a,\alpha)$ for a's *far from* 1 and on the other hand for a's *close to* 1 we use some techniques involving Fourier coefficients and estimates of hypergeometric functions. We begin with a lemma which gives an upper estimate for $F_j(a,\alpha)$ in terms of $\|f\|_\infty$, a and j.

Lemma 2.1. *If $f:[0,2\pi] \to \mathbb{R}$ is continuous, then for every $j \in (0,2)$ and every $a \in (1,+\infty)$,*

$$F_j(a,\alpha) \leqslant \|f\|_\infty \Theta_j(a) = \|f\|_\infty a^j \left(\frac{\log(a^2+\sqrt{a^4-1})}{\sqrt{a^4-1}}\right)^{\min\{j,1\}}$$

and $F_j(a,\alpha) \leqslant \|f\|_\infty \Theta_j(1/a)$ for all $0<a<1$.

Proof. We can assume without loss of generality that $\|f\|_\infty = 1$, which implies that

$$F_j(a,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta-\alpha)\,d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} \leqslant \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}}.$$

If we take $a>1$ and $j>1$ we obtain that

$$\begin{aligned}
\phi(a) &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} = \frac{2a^j}{\pi}\int_0^{\pi/2} \frac{d\theta}{((a^4-1)\cos^2\theta+1)^{j/2}} \\
&\leqslant \frac{2a^j}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{(a^4-1)\cos^2\theta+1}} \leqslant \frac{2a^j}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{((1-2\theta/\pi)\sqrt{a^4-1}\,)^2+1}} \\
&= \frac{a^j}{\sqrt{a^4-1}}\log\bigl(a^2+\sqrt{a^4-1}\,\bigr).
\end{aligned}$$

On the other hand if we take $a > 1$ and $0 < j < 1$ we use Jensen inequality and then we arrive at

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} \leqslant \left(\frac{a}{\sqrt{a^4-1}}\log\bigl(a^2+\sqrt{a^4-1}\,\bigr)\right)^j.$$

If $a < 1$, notice that $\phi(a) = \phi(1/a)$, and hence the result follows from the case $a > 1$. □

This result proves Theorem 1.4 for a's *large enough*, as the following corollary shows.

Corollary 2.2. *Let $f : \mathbb{T} \to (0, \infty)$ be as in Theorem* 1.4*. Then for every $\alpha \in \mathbb{R}$ and every $a \geqslant 5.686$,*

$$F_1(1,\alpha) \leqslant F_1(1,0).$$

Proof. If $a \geqslant 5.686$ it can be checked that

$$\frac{a}{\sqrt{a^4-1}}\log\bigl(a^2+\sqrt{a^4-1}\,\bigr) < 0.734,$$

hence by using Lemma 2.1, since $\|f\|_\infty < 1.261B_0$, we get that for every $\alpha \in \mathbb{R}$ and every $a \geqslant 5.686$,

$$F_1(a,\alpha) \leqslant 0.734\|f\|_\infty < 0.926B_0 < B_0 = F_1(1,0).$$

(The numerical computations have been performed with Maple processor.) □

As a consequence of the last lemma and corollary the only thing we have to do to complete the proof of the main theorem is to prove the inequality in the range of a's for which $\Theta_1(a) > 1.261^{-1}$ (i.e., close to $a = 1$).

In order to study the situation for a's *close to* 1, let us introduce some notation. If $a > 0$ and $j \in (0, 2)$ we define $g_a(\theta)$ for every $\theta \in \mathbb{T}$ by

$$g_a(\theta) = \frac{1}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}}.$$

The following lemma allows us study the inequality (1.3) in terms of Fourier coefficients.

Lemma 2.3. *Let g_a be defined as before. If we denote by*

$$\ell = \frac{a^2-1}{a^2+1} \in (-1, 1), \tag{2.4}$$

then

$$g_a(\theta) = \sum_{k=0}^{\infty} A_{2k}\cos(2k\theta),$$

where

$$A_0 = (1-\ell^2)^{j/2} \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}^2, \tag{2.5}$$

$$A_{2k} = 2\ell^k (1-\ell^2)^{j/2} \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m} \binom{-j/2}{m+k}, \tag{2.6}$$

and the trigonometric series converges absolutely and uniformly in θ. Furthermore $A_0 < 1$ whenever $a \neq 1$ and

(i) $\{A_{2k}\}_{k=0}^{\infty}$ *is a nonincreasing sequence convergent to* 0 *if* $a < 1$,
(ii) $\{(-1)^k A_{2k}\}_{k=0}^{\infty}$ *is a nonincreasing sequence convergent to* 0 *otherwise.*

Proof. It is very easy to see that for every $\theta \in \mathbb{T}$,

$$\begin{aligned} \frac{1}{a^2\cos^2\theta + a^{-2}\sin^2\theta} &= \frac{1-\ell^2}{(1+\ell)^2\cos^2\theta + (1-\ell)^2\sin^2\theta} \\ &= \frac{1-\ell^2}{|e^{i\theta} + \ell e^{-i\theta}|^2} = \frac{1-\ell^2}{(1+\ell e^{-2i\theta})(1+\ell e^{2i\theta})}. \end{aligned}$$

So for every $\theta \in \mathbb{T}$,

$$\begin{aligned} (1-\ell^2)^{-j/2} g_a(\theta) &= \sum_{n,m=0}^{\infty} \binom{-j/2}{n}\binom{-j/2}{m} \ell^{n+m} e^{2i\theta(n-m)} \\ &= \sum_{m=0}^{\infty} \sum_{k=-m}^{\infty} \binom{-j/2}{m}\binom{-j/2}{m+k} \ell^{2m+k} e^{2i\theta k} \\ &= \sum_{k=0}^{\infty} \ell^k e^{2i\theta k} \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}\binom{-j/2}{m+k} \\ &\quad + \sum_{k=-\infty}^{-1} \ell^k e^{2i\theta k} \sum_{m=-k}^{\infty} \ell^{2m} \binom{-j/2}{m}\binom{-j/2}{m+k} \\ &= \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}^2 + 2\sum_{k=1}^{\infty} \ell^k \cos(2k\theta) \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}\binom{-j/2}{m+k} \end{aligned}$$

and we get (2.5) and (2.6).

Since $0 < j < 2$, we get that

$$\frac{j/2+m+k}{m+k+1} < 1$$

and

$$\left| \binom{-j/2}{m}\binom{-j/2}{m+k} \right| = (-1)^{2m+k} \binom{-j/2}{m}\binom{-j/2}{m+k}.$$

Hence

$$\left|\binom{-j/2}{m}\binom{-j/2}{m+k}\right| \geqslant \left|\binom{-j/2}{m}\binom{-j/2}{m+k+1}\right|,$$

which implies the monotonic character stated in (i) and (ii). Eventually, since the function $h(t)=t^{j/2}$ is concave in $[0,+\infty)$, we get that whenever $a\neq 1$,

$$A_0=\frac{1}{2\pi}\int_0^{2\pi} g_a(\theta)\,d\theta < \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a^j\cos^2\theta+a^{-j}\sin^2\theta}=1. \qquad \square$$

We come back to the proof of the theorem. According to the preceding lemma, in order to prove Theorem 1.4 it is enough to show that

$$F_j(a,\alpha)=\sum_{k=0}^{\infty} A_{2k}\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos 2k(\theta+\alpha)\,d\theta \leqslant B_0=F_j(1,0)$$

and so, if we assume the conditions imposed on the function f in Theorem 1.4 the problem reduces to showing that

$$\sum_{k=0}^{\infty} A_{4k}B_{4k}\cos(4\alpha)\leqslant B_0 \tag{2.7}$$

for all $\ell\in(-1,1)$ and $\alpha\in\mathbb{R}$ (recall that A_k's depend on ℓ). Hence, we have to obtain sharp estimates for A_{4k}.

Note that the A_{4k} are hypergeometric functions. Indeed, given $a,b,c\in\mathbb{R}$ and $z\in\mathbb{C}$ the *hypergeometric function* $F(a,b;c;z)$ is defined by

$$F(a,b;c;z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!},$$

where $(a)_m=a(a+1)\dots(a+m-1)$ (see [1]). It can be checked that

$$\begin{aligned}
A_0&=(1-\ell^2)^{j/2}F\left(\frac{j}{2},\frac{j}{2};1;\ell^2\right),\\
A_{4k}&=2\ell^{2k}(1-\ell^2)^{j/2}\binom{-j/2}{2k}F\left(\frac{j}{2},\frac{j}{2}+2k;2k+1;\ell^2\right)
\end{aligned}$$

for $k\geqslant 1$. In order to get some upper estimates for A_{4k} that will be useful later, we give some general upper estimates for some hypergeometric functions.

Lemma 2.4. *Let $\alpha\in(0,1)$ and $k\in\mathbb{N}\cup\{0\}$. For every $x\in(-1,1)$,*

$$\begin{aligned}
&F(\alpha,\alpha;1;x)\leqslant e^{\alpha-1}(1-x)^{-\alpha}+(1-e^{\alpha-1})+x(\alpha^2-\alpha e^{\alpha-1}),\\
&F(\alpha,\alpha+2k;2k+1;x)\\
&\quad\leqslant\frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}}\left[(-1)^k\binom{-\alpha}{k}e^{(\alpha-1)/2}\right.
\end{aligned}$$

$$- (1-x)^{\alpha}\left((-1)^k\binom{-\alpha}{k}e^{(\alpha-1)/2} - \binom{-\alpha}{2k}\right)\Bigg]$$
$$\leqslant \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}}(-1)^k\binom{-\alpha}{k}e^{(\alpha-1)/2}.$$

Proof. First of all, we study $F(\alpha,\alpha+2k;2k+1;x)$. If $m \geqslant 0$ and $k \geqslant 1$, on the one hand

$$0 \leqslant \frac{\alpha\ldots(\alpha+m-1)}{m!} = (-1)^m\binom{-\alpha}{m}.$$

On the other hand

$$\begin{aligned}
\frac{(\alpha+2k)_m}{(2k+1)_m} &= \frac{(\alpha+2k)\ldots(\alpha+2k+m-1)}{(2k+1)\ldots(2k+m)} \\
&= \frac{(2k)!}{(\alpha)\ldots(\alpha+2k-1)}\,\frac{\alpha(\alpha+1)\ldots(\alpha+2k+m-1)}{(2k+m)!} \\
&= \binom{-\alpha}{2k}^{-1}\frac{\alpha\ldots(\alpha+k-1)}{k!}\,\frac{(\alpha+k)\ldots(\alpha+2k+m-1)}{(k+1)\ldots(2k+m)} \\
&= (-1)^k\binom{-\alpha}{k}\binom{-\alpha}{2k}^{-1}\prod_{n=k+1}^{2k+m}\frac{\alpha+n-1}{n}.
\end{aligned}$$

Hence, since $\alpha < 1$, we get that

$$\begin{aligned}
\frac{(\alpha+2k)_m}{(2k+1)_m} &\leqslant (-1)^k\binom{-\alpha}{k}\binom{-\alpha}{2k}^{-1}\left(\frac{\alpha+2k+m-1}{2k+m}\right)^{m+k} \\
&\leqslant (-1)^k\binom{-\alpha}{k}\binom{-\alpha}{2k}^{-1}e^{(\alpha-1)/2}.
\end{aligned}$$

Then

$$\begin{aligned}
F(\alpha,\alpha+2k;2k+1;x) &= \sum_{m=0}^{\infty}\frac{(\alpha)_m(\alpha+2k)_m}{m!(2k+1)_m}x^m \\
&\leqslant (-1)^k\binom{-\alpha}{k}\binom{-\alpha}{2k}^{-1}e^{(\alpha-1)/2}\sum_{m=0}^{\infty}\binom{-\alpha}{m}(-x)^m \\
&= \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}}(-1)^k\binom{-\alpha}{k}e^{(\alpha-1)/2}
\end{aligned}$$

and also

$$\begin{aligned}
F(\alpha,\alpha+2k;2k+1;x) &= 1+\sum_{m=1}^{\infty}\frac{(\alpha)_m(\alpha+2k)_m}{m!(2k+1)_m}x^m \\
&\leqslant 1+(-1)^k\binom{-\alpha}{k}\binom{-\alpha}{2k}^{-1}e^{(\alpha-1)/2}\sum_{m=1}^{\infty}\binom{-\alpha}{m}(-x)^m
\end{aligned}$$

$$= \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}} \left[(-1)^k \binom{-\alpha}{k} e^{(\alpha-1)/2} - (1-x)^{\alpha} \left((-1)^k \binom{-\alpha}{k} e^{(\alpha-1)/2} - \binom{-\alpha}{2k} \right) \right].$$

In order to get upper estimates for $F(\alpha, \alpha; 1; x)$, we compute $(\alpha)_m^2/((1)_m m!)$. If $m \geqslant 0$,

$$\frac{(\alpha)_m^2}{(1)_m m!} = (-1)^m \binom{\alpha}{m} \prod_{n=1}^{m} \left(\frac{\alpha + n - 1}{n} \right) \leqslant (-1)^m \binom{\alpha}{m} \left(\frac{\alpha + m - 1}{m} \right)^m = (-1)^m \binom{\alpha}{m} e^{\alpha - 1}$$

and we proceed as before. □

Corollary 2.5. *If $j = 1$ and under the same conditions as in Lemma 2.3, then*

$$A_0 \leqslant e^{-1/2} + (1-\ell^2)^{1/2} \left[1 - e^{-1/2} - \frac{\ell^2}{2} \left(e^{-1/2} - \frac{1}{2} \right) \right],$$

$$A_4 \leqslant e^{-1/4} \ell^2 \left[1 - (1-\ell^2)^{1/2} \left(1 - \frac{3e^{1/4}}{4} \right) \right],$$

$$A_{4k} \leqslant 2e^{-1/4} \ell^{2k} (-1)^k \binom{-1/2}{k}, \quad k \geqslant 1,$$

$$\sum_{k=2}^{\infty} A_{4k} \leqslant 2e^{-1/4} \left((1-\ell^2)^{-1/2} - 1 - \frac{\ell^2}{2} \right).$$

These upper estimates allow us to continue with the proof of the theorem. By using the preceding lemmas and the estimates for the coefficients A_{4k} $(k = 0, \ldots)$ for $j = 1$, we get that

$$\begin{aligned}
\sum_{k=0}^{\infty} A_{4k} B_{4k} \cos(4\alpha) &\leqslant \sum_{k=0}^{\infty} A_{4k} |B_{4k}| \\
&\leqslant B_0 \left(A_0 + \frac{|B_4|}{B_0} A_4 + \frac{|B_8|}{B_0} \sum_{k=2}^{\infty} A_{4k} \right) \\
&\leqslant B_0 \left(A_0 + 0.070 A_4 + 0.022 \sum_{k=2}^{\infty} A_{4k} \right) \\
&\leqslant B_0 \Bigg[e^{-1/2} - 0.044 e^{-1/4} + \sqrt{1-\ell^2} (1 - e^{-1/2}) + 0.048 e^{-1/4} \ell^2 \\
&\quad - \frac{1}{2} \ell^2 \sqrt{1-\ell^2} (e^{-1/2} + 0.14 e^{-1/4} - 0.605) + \frac{0.044 e^{-1/4}}{\sqrt{1-\ell^2}} \Bigg] \\
&< B_0,
\end{aligned}$$

whenever $0 < \ell^2 \leqslant 0.89$. Indeed, if we take the function ψ defined for every $x \in [0, 1)$ by

$$\psi(x) = e^{-1/2} - 0.044e^{-1/4} + (1 - e^{-1/2})\sqrt{1-x} + 0.048e^{-1/4}x$$
$$- \frac{1}{2}x\sqrt{1-x}(e^{-1/2} + 0.14e^{-1/4} - 0.605) + \frac{0.044e^{-1/4}}{\sqrt{1-x}},$$

a direct computation shows that

$$\psi'(x) \leqslant \frac{-0.196 + 0.028x}{\sqrt{1-x}} + 0.038 - 0.055\sqrt{1-x} + \frac{0.018}{(1-x)^{3/2}} < 0$$

for all $x < 0.89$. Since $\psi(0) < 1$ we achieve that $\psi(\ell^2) < 1$ whenever $0 < \ell < 0.943$ which implies that $F_1(a,\alpha) \leqslant F_1(1,0)$ for all $\alpha \in \mathbb{R}$ and $1 < a < 5.686$.

If $a \geqslant 5.686$, the result follows from Lemma 2.1 and Corollary 2.2.

Remark 2.6. We might extend Theorem 1.4 to more general functions $f : \mathbb{T} \to (0, +\infty)$ with enough symmetries whose Fourier coefficients satisfy weaker conditions than those appearing in Theorem 1.4 (condition (i)), simply by considering sharper estimates for the hypergeometric functions in Lemma 2.4. These estimates could be easily established by the technique used in that lemma, simply by considering sharper expressions for $F(\alpha, \alpha + 2k; 2k + 1; x)$.

3. An application to the extreme dual quermassintegral of convex bodies

According to Section 1, the motivation of Theorem 1.4 comes from the characterization of extreme dual 'quermassintegrals' of a convex body in terms of isotropic measures.

In this section we will consider a positive continuous function $f : \mathbb{T} \to (0, +\infty)$ such that for every $\theta \in \mathbb{T}$,

$$f(\theta) = f(2\pi - \theta) = f(\pi - \theta) = f\left(\frac{\pi}{2} - \theta\right).$$

In this case we will say that f has "*enough symmetries*." Examples of the kind of functions that we have in mind are the functions of the form $f(\cdot) = \rho_K(\cdot)^{2-j}$, where K is a convex body symmetric with respect to the axes and the bisectors. It is easy to check that if $\rho_K(\cdot)^{2-j}$ has "*enough symmetries*" then $\rho_K(\cdot)^{2-j}\,d\sigma(\cdot)$ is isotropic in $\mathbb{T}$. For every $k \in \mathbb{Z}$, we will define B_k by

$$B_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)\,d\theta.$$

The following lemmas study the behaviour of the Fourier coefficients B_k of functions with "*enough symmetries*."

Lemma 3.1. *Let $f : \mathbb{T} \to (0, +\infty)$ be a function with "enough symmetries." If f' is nonpositive and nondecreasing on $[0, \pi/4]$, then $B_{4k} \geqslant 0$ for all $k \geqslant 0$.*

Proof. By the symmetries of f and integrating by parts it is easy to show that

$$B_{4k} = \frac{4}{\pi} \int_0^{\pi/4} f'(\theta) \sin(4k\theta)\, d\theta.$$

If $k \geqslant 1$

$$\int_0^{\pi/4} f'(\theta)\sin(4k\theta)\,d\theta = \int_0^{\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta + \int_{\pi/4k}^{2\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta + \cdots + \int_{(k-1)\pi/4k}^{k\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta.$$

Since f' is nondecreasing on $[0, \pi/4]$, we deduce that

$$\int_0^{\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta + \int_{\pi/4k}^{2\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta = \int_0^{\pi/4k} \left(f'(\theta) - f'\left(\theta + \frac{\pi}{4k}\right)\right)\sin(4k\theta)\,d\theta \leqslant 0.$$

In fact, by using the same idea, for every $i = 1, \dots, [k/2]$ we get that

$$\int_{(2i-2)\pi/4k}^{(2i-1)\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta + \int_{(2i-1)\pi/4k}^{2i\pi/4k} f'(\theta)\sin(4k\theta)\,d\theta \leqslant 0,$$

hence

$$\int_0^{\pi/4} f'(\theta)\sin(4k\theta)\,d\theta \leqslant 0,$$

if k is even. In the other case, the last summand is also negative since $f' \leqslant 0$ and $\sin(4k\theta) \geqslant 0$ in that interval. Eventually, we get that $B_{4k} \geqslant 0$ for all $k \geqslant 1$ and therefore the result holds. □

Lemma 3.2. *Let f be, as before, a continuous positive function with "enough symmetries." If $F(\theta) = f(\pi/4 - \theta)\cos(2\theta)$ is such that $F'(0) = 0$ and the function $F''(\theta)$ is nonpositive and nonincreasing in $[0, \pi/4]$, then $B_{4k} \geqslant B_{4k+4}$ for all $k \geqslant 0$.*

Proof. By changing variables we get that

$$
\begin{aligned}
B_{4k} - B_{4k+4} &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\big[\cos(4k\theta) - \cos\big((4k+4)\theta\big)\big]\, d\theta \\
&= \frac{(-1)^k}{2\pi} \int_0^{2\pi} f\left(\frac{\pi}{4} - \theta\right)\big[\cos(4k\theta) + \cos\big((4k+4)\theta\big)\big]\, d\theta \\
&= \frac{(-1)^k}{\pi} \int_0^{2\pi} f\left(\frac{\pi}{4} - \theta\right) \cos\big((4k+2)\theta\big)\cos(2\theta)\, d\theta \\
&= (-1)^k \frac{8}{\pi} \int_0^{\pi/4} F(\theta)\cos\big((4k+2)\theta\big)\, d\theta.
\end{aligned}
$$

Now, by integrating twice by parts,

$$
B_{4k} - B_{4k+4} = -\frac{8(-1)^k}{\pi(4k+2)^2} \int_0^{\pi/4} F''(\theta)\cos\big((4k+2)\theta\big)\, d\theta.
$$

Let $k = 2m$ be an even number. Then

$$
\begin{aligned}
B_{4k} - B_{4k+4} = \frac{8}{\pi(4k+2)^2} \Bigg(& \int_0^{\pi/2(4k+2)} -F''(\theta)\cos\big((4k+2)\theta\big)\, d\theta \\
& + \int_{\pi/2(4k+2)}^{5\pi/2(4k+2)} -F''(\theta)\cos\big((4k+2)\theta\big)\, d\theta \\
& + \cdots + \int_{(4m-3)\pi/2(4k+2)}^{(4m+1)\pi/2(4k+2)} -F''(\theta)\cos\big((4k+2)\theta\big)\, d\theta \Bigg).
\end{aligned}
$$

Since $-F'' \geqslant 0$ and $-F''$ is nondecreasing on $[0, \pi/4]$ we obtain that

$$
\int_0^{\pi/2(4k+2)} -F''(\theta)\cos\big((4k+2)\theta\big)\, d\theta \geqslant 0
$$

and for every $i = 1, \ldots, m$,

$$
\int_{(4i-3)\pi/2(4k+2)}^{(4i+1)\pi/2(4k+2)} -F''(\theta)\cos\big((4k+2)\theta\big)\, d\theta \geqslant 0,
$$

which ensures that $B_{4k} \geqslant B_{4k+4}$.

Let now $k = 2m + 1$ be an odd number. As before,

$$
\begin{aligned}
B_{4k} - B_{4k+4} = -\frac{8}{\pi(4k+2)^2}\Biggl(& \int_0^{3\pi/2(4k+2)} -F''(\theta)\cos\bigl((4k+2)\theta\bigr)\,d\theta \\
& + \int_{3\pi/2(4k+2)}^{7\pi/2(4k+2)} -F''(\theta)\cos\bigl((4k+2)\theta\bigr)\,d\theta \\
& + \cdots + \int_{(4m-1)\pi/2(4k+2)}^{(4m+3)\pi/2(4k+2)} -F''(\theta)\cos\bigl((4k+2)\theta\bigr)\,d\theta \Biggr).
\end{aligned}
$$

By similar reasons as before we get that

$$\int_0^{3\pi/2(4k+2)} -F''(\theta)\cos\bigl((4k+2)\theta\bigr)\,d\theta \leqslant 0$$

and for every $i = 1, \ldots, m$,

$$\int_{(4i-1)\pi/2(4k+2)}^{(4i+3)\pi/2(4k+2)} -F''(\theta)\cos\bigl((4k+2)\theta\bigr)\,d\theta \leqslant 0,$$

and so the conclusion of the lemma holds. $\square$

The previous lemmas allow us to get some information about the Fourier coefficients of

$$f(\theta) = \rho_{B_1^2}^{2-j}(\theta) = \frac{1}{(|\sin\theta| + |\cos\theta|)^{2-j}}$$

and we obtain the following result.

Proposition 3.3. *Let* $j \in (0, x_0]$ ($x_0 = -5/3 + \sqrt{73}/3 \simeq 1.18$). *Then the Fourier coefficients* B_{4k} *of* $f(\cdot) = \rho_{B_1^2}^{2-j}(\cdot)$ *are such that* $B_0 \geqslant B_4 \geqslant \cdots \geqslant B_{4k} \geqslant \cdots \geqslant 0$.

Furthermore since for $j = 1$ we can compute the Fourier coefficients of $\rho_{B_1^2}$, by using Maple we get that $B_0 = 0.793515\ldots$, $B_4 = 0.055311\ldots$, $B_8 = 0.017445\ldots$ and $\|\rho_{B_1^2}\|_\infty = 1$. Therefore, we can apply Theorem 1.4 and we conclude that

$$\tilde{W}_1(B_1^2) = \max\bigl\{W_1(SB_1^2);\ S \in SL(2)\bigr\}.$$

Remark 3.4. This technique could be also applied to other "*symmetric enough*" convex bodies K, simply by considering the estimates given in Corollary 2.5 properly improved, as we noticed in Remark 2.6, provided that we could get some control on the behaviour of the Fourier coefficients of $f(\cdot) = \rho_K^{2-j}(\cdot)$. This might be useful, for example, for other ℓ_p^2 balls. If we would like to obtain results in $\mathbb{R}^n$ ($n > 2$), we should use spherical harmonics instead of Fourier coefficients.

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